

MAXIMIZING ORBITS FOR HIGHER DIMENSIONAL CONVEX BILLIARDS

MISHA BIALY

ABSTRACT. The main result of this paper is, that for convex billiards in higher dimensions, in contrast with 2D case, for every point on the boundary and for every n there always exist billiard trajectories developing conjugate points at the n -th collision with the boundary. We shall explain that this is a consequence of the following variational property of the billiard orbits in higher dimension. If a segment of an orbit is locally maximizing, then it can not pass too close to the boundary. This fact follows from the second variation formula for the Length functional. It turns out that this formula behaves differently with respect to "longitudinal" and "transversal" variations.

1. INTRODUCTION AND BACKGROUND

This paper is motivated by a question by Jürgen Moser about conjugate points for Birkhoff billiards in dimensions higher than 2. The main result of this paper is that in higher dimensions, in contrast with 2D case, for every point on the boundary and for every n there always exist billiard trajectories developing conjugate points precisely at the n -th collision with the boundary. This fact, as we shall see, is a consequence of the following variational property of the billiard orbits in higher dimension. There are no locally maximizing configurations, passing too close to the boundary. This follows from the second variation formula for the Length functional. It turns out that this formula behaves differently with respect to "longitudinal" and "transversal" variations. As an example of this calculation let me mention the example of standard sphere where the only orbits which are locally maximizing are that of the diameters.

A natural extension of Moser's question is, if there are variational properties of billiard configurations distinguishing standard sphere among other bodies. This remains still open, though we formulate a conjecture in this direction below.

Let me denote by Σ a strictly convex hypersurface in the Euclidean space \mathbb{R}^d , we shall assume throughout this paper, that it is at least C^2 -smooth with strictly positive curvature at every point. Any billiard configuration determines a sequence of collision points with the

Date: 23 July 2008.

boundary, $\{x_n\}_{n \in \mathbb{Z}}, x_n \in \Sigma$. Billiard configurations are in one to one correspondence with the critical points of the functional

$$\Phi\{u_n\} = \sum_{n \in \mathbb{Z}} L(u_n, u_{n+1}),$$

where

$$L : \Sigma \times \Sigma \rightarrow \mathbb{R}, L(x, y) = |x - y|,$$

denotes the Euclidean distance function in \mathbb{R}^d . More precisely, this correspondence means, that any finite segment of trajectory $\{x_n\}_{n \in [M, N]}$ is the critical point of the function

$$\Phi_{M, N}(u_M, \dots, u_N) = L(x_{M-1}, u_M) + \sum_{i=M}^{N-1} L(u_i, u_{i+1}) + L(u_N, x_{N+1}).$$

Notice that L has singularities on the diagonal which complicate the variational analysis of the critical points of the functional Φ and $\Phi_{M, N}$. In this paper we shall adopt the following:

- Definition 1.1.** (a) The segment of billiard orbit $\{x_n\}_{n \in [M, N]}$ is called maximizing if is a local maximum for the function $\Phi_{M+1, N-1}$.
 (b) An infinite orbit is called maximizing if any finite segment of it is maximizing.
 (c) The sequence of tangent vectors $\xi_n \in T_{x_n} \Sigma$ at the points of a billiard configuration $\{x_n\}$ is called Jacobi field it appears as a variation field, $\xi_n = \frac{d}{d\varepsilon} \big|_{\varepsilon=0} (x(\varepsilon)_n)$ for a variation of the initial configuration.
 (d) Two points x_M, x_N of the billiard ball configuration are called conjugate if there is a non zero Jacobi field vanishing at x_M, x_N .

Let me remark that the last definition of conjugate points has very clear meaning on the language of geometric optics. It means that for the light ray starting at x_M becomes focused at x_N , so in other words it is a "bright" point on the boundary.

For the plane convex billiards there are lot of maximizing billiard configurations. They appear in the so called Aubry -Mather sets. For example any billiard configuration tangent to a smooth convex caustic is maximizing (notice that by KAM type theorem of Lazutkin ([8]), there are infinitely many convex caustics near the boundary, for any sufficiently smooth convex billiard table). On the other hand, I have proved in ([3]) that only for circles all configurations are maximizing, in other words for any non circular billiard there always exist conjugate points. In contrast with this in higher dimensions for any shape with no exceptions there always exist conjugate points and in fact many of them.

Theorem 1.2. *Let $\Sigma \subset \mathbb{R}^d, d > 2$ be any C^2 -smooth strictly convex hypersurface with positive curvature. Then for any point $x \in \Sigma$ there always exist conjugate points along infinitely many configurations starting at x .*

Theorem 1.3. *There exists a constant $C(\Sigma)$ such that for any maximizing configuration $\{x_n\}$ the angle of reflections φ_n at any vertex x_n can not be too small:*

$$\varphi_n > C(\Sigma)$$

Let me come back to the question by J.Moser. Analyzing the proofs of the Theorems 1.3 and, 1.2 below, one naturally comes to the following conjectures. Conjecturally there are no convex hypersurfaces different from spheres, with the property that all billiard orbits are maximizing with respect to "longitudinal" perturbations. A somewhat related version of this conjecture is the following: there are no convex hypersurfaces different from spheres such that every billiard trajectory has a non-vanishing longitudinal Jacobi field. Let me remark, that the application of my previous ideas from ([3]) lead naturally to some new integral geometric quantities measuring certain "roundedness" for bodies of constant width, and interesting inequalities of isoperimetric type. However, I can not claim for the moment that these inequalities distinguish spheres among other convex bodies. Let me mention also that, if the conjectures are true they could be considered as a kind of "non-holonomic" version of a theorem by R Sine, saying that the spheres are the only convex hypersurfaces with the property that any billiard configuration lies in a 2-plane (see [10] for the proof and discussions). I hope to discuss these questions elsewhere.

Let me finish this introduction with the following remark. It is in fact an old idea going back at least to Hedlund and Morse, and alive till nowadays, to construct invariant sets of Hamiltonian systems by variational methods. However it is always a problem to decide if such a set has zero, positive or full measure in the phase space. Let me give here two examples: for planar billiards the set of maximizing orbits has positive measure, by the result of [8], mentioned above. It was proved in [3], that the set of maximizing orbits can not occupy the set of full measure in the phase space. It is a very interesting question what happens in this perspective for convex billiards of higher dimension. What can be said about the set of maximizing orbits or about the set of maximizing with respect to longitudinal perturbations. Let me remark also, that it was proved by M.Berger that there are no convex caustics in higher dimensional billiards except for the ellipsoids (see [2]).

In the next two section we shall derive the computation of second derivatives of the function L and the second variation formulas. This will be crucial, in fact, for the proof of the main theorems. Then we shall provide examples in Section 4, and prove the existence of conjugate points in Section 5.

ACKNOWLEDGEMENTS

It is my pleasure to thank Misha Sodin and Serezha Tabachnikov for useful and stimulating discussions.

2. DERIVATIVES OF THE DISTANCE FUNCTION L

Let me introduce first the Billiard ball map. This is symplectic diffeomorphism of the unit ball cotangent bundle of Σ for which L serves as a generating function. Everywhere in this paper we will identify the cotangent and tangent bundles of Σ with the help of Riemannian metric induced on Σ . As usual for any point $x \in \Sigma$ and for a inward unit vector $z \in T_x \mathbb{R}^d$ one can define a vector $v \in B_x \Sigma, |v| < 1$ by the projection π_x of z along the normal n_x so that,

$$v = z - \hat{v}n_x \text{ where } \hat{v} = \sqrt{1 - v^2} = \sin \varphi,$$

and ϕ stands for the angle between z and $T_x \Sigma$. By means of these notations the billiard ball map $T : B^* \Sigma \rightarrow B^* \Sigma$ is defined implicitly by the following

$$T : (x, v) \mapsto (y, w) \Leftrightarrow$$

$$(1) \quad L_1(x, y) = -\pi_x\left(\frac{y - x}{|y - x|}\right) = -v, \quad L_2(x, y) = \pi_y\left(\frac{y - x}{|y - x|}\right) = w$$

Here L_1, L_2 denote the (co)vectors, which are differentials of L with respect to x, y respectively. Notice that the formulas (1) mean precisely that the orbits of T are in one to one correspondence with the extremals of Φ which are precisely the billiard configurations. In fact the calculation of the mixed derivative below (the so called twist condition) imply that T is a genuine diffeomorphism. These formulas for the second partial derivatives of L can be obtained by a direct calculation.

Proposition 2.1. *For any two distinct points $x, y \in \Sigma$ the linear operators $L_{11}, L_{12}, L_{21}, L_{22}$ act by the following formulas:*

$$(a) \quad L_{11}(x, y) : T_x \Sigma \rightarrow T_x \Sigma, \quad L_{11}(\xi) = (\xi - \langle v, \xi \rangle v) / L - S(\xi) \hat{v},$$

$$(b) \quad L_{22}(x, y) : T_y \Sigma \rightarrow T_y \Sigma, \quad L_{22}(\eta) = (\eta - \langle w, \eta \rangle w) / L - S(\eta) \hat{w},$$

$$(c) \quad L_{12}(x, y) : T_y \Sigma \rightarrow T_x \Sigma, \quad L_{12}(\eta) = (-\pi_x(\eta) + \langle w, \eta \rangle v) / L$$

$$(d) \quad L_{21}(x, y) : T_x \Sigma \rightarrow T_y \Sigma, \quad L_{21}(\xi) = (-\pi_y(\xi) + \langle v, \xi \rangle w) / L$$

for any $\xi \in T_x \Sigma, \eta \in T_y \Sigma$. Here S denotes the shape operator of Σ : $S(\xi) = -\nabla_\xi n_x$. Moreover the operators L_{12} and L_{21} are isomorphisms which are adjoint one to the other.

Remark 1. The last property of L_{12}, L_{21} being isomorphisms is the so called twist condition, replacing the Legendre condition of calculus of variations for continuous time.

Proof. We shall derive (a) first and then (d), all other formulas are analogous. For (a) y is fixed and x varies in the direction of $\xi \in T_x \Sigma$. Denote by $\nabla, \tilde{\nabla}$ the standard connections on Σ and \mathbb{R}^d respectively. We have:

$$\begin{aligned}
L_{11}(\xi) &= \nabla_\xi L_1(x, y) = \nabla_\xi \left(\frac{x-y}{L} - \left\langle \frac{x-y}{L}, n_x \right\rangle n_x \right) = \\
&= \pi_x \tilde{\nabla}_\xi \left(\frac{x-y}{L} - \left\langle \frac{x-y}{L}, n_x \right\rangle n_x \right) = \\
&= \pi_x \left(\frac{\xi}{L} - \frac{x-y}{L^2} \left\langle L_1, \xi \right\rangle - \left\langle \frac{x-y}{L}, n_x \right\rangle \nabla_\xi n_x \right) = \\
&= \frac{\xi}{L} - \frac{\left\langle L_1, \xi \right\rangle}{L} L_1 + \left\langle L_1, n_x \right\rangle S(\xi) = \\
&= \frac{\xi}{L} - \frac{\left\langle v, \xi \right\rangle}{L} v - S(\xi) \hat{v}
\end{aligned}$$

In order to prove (d) let $\gamma(t)$ be any curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Then we have

$$\begin{aligned}
L_{21}(\xi) &= \frac{d}{dt} \Big|_{t=0} \left(\frac{y-x}{L} - \frac{\left\langle y-x, n_y \right\rangle}{L} n_y \right) = \\
&= \pi_y \left(-\frac{\xi}{L} - \frac{y-x}{L^2} \left\langle L_1, \xi \right\rangle \right) = \\
&= -\pi_y \left(\frac{\xi}{L} \right) - \frac{L_2}{L} \left\langle L_1, \xi \right\rangle = -\frac{\xi}{L} + \left\langle \frac{\xi}{L}, n_y \right\rangle n_y + \frac{w}{L} \left\langle v, \xi \right\rangle
\end{aligned}$$

□

3. SECOND VARIATION FORMULAS

Let $\{x_n\}_{n \in \mathbb{Z}}$ be a billiard configuration. Pick $M \leq N$ and the tangent vectors $\xi_n \in T_{x_n} \Sigma$, $n \in [M, N]$. With the help of the operators of second partials the quadratic form of the second variation for the functional $\Phi_{M,N}$ is the following:

$$\begin{aligned}
(2) \quad \delta^2 \Phi_{MN}(\xi_M, \dots, \xi_N) &= \sum_{n=M}^N \left\langle (L_{11}(x_n, x_{n+1}) + L_{22}(x_{n-1}, x_n)) \xi_n, \xi_n \right\rangle + \\
&\quad + 2 \sum_{n=M}^{N-1} \left\langle L_{12}(x_n, x_{n+1}) \xi_{n+1}, \xi_n \right\rangle.
\end{aligned}$$

Let me apply now the formulas of the Proposition in a very special case, the orbit of one reflection, $\{x_{-1}, x_0, x_1\}$, i.e. $M = 0, N = 0$. We have

$$(3) \quad \delta^2 \Phi_{00}(\xi_0) = (\xi_0^2 - \left\langle v_0, \xi_0 \right\rangle^2) \left(\frac{1}{L(x_{-1}, x_0)} + \frac{1}{L(x_0, x_1)} \right) - 2B(\xi_0, \xi_0) \hat{v}_0,$$

where $B(\xi_0, \xi_0) = \langle S(\xi_0), \xi_0 \rangle$ is the second fundamental form, and as before $v_0 = \pi_{x_0}((x_1 - x_0)/L(x_0, x_1))$ and $\hat{v}_0 = \sqrt{1 - v_0^2} = \sin \varphi_0$ is the sinus of the angle of reflection. Notice that this formula gives different answers for the vectors ξ_0 orthogonal or parallel to v_0 , as follows:

$$(4) \quad \delta^2 \Phi_{00}(\xi) = \xi^2 \left(\frac{1}{L(x_{-1}, x_0)} + \frac{1}{L(x_0, x_1)} \right) - 2B(\xi, \xi) \hat{v}_0, \quad \text{for } \xi \perp v_0,$$

$$(5) \quad \delta^2 \Phi_{00}(\xi) = \xi^2 \hat{v}_0^2 \left(\frac{1}{L(x_{-1}, x_0)} + \frac{1}{L(x_0, x_1)} \right) - 2B(\xi, \xi) \hat{v}_0, \quad \text{for } \xi \parallel v_0$$

Using these formulas we obtain the following

Theorem 3.1. *There exists a constant $C(\Sigma) > 0$, such that for any piece of billiard trajectory $\{x_{-1}, x_0, x_1\}$ having angle of reflection φ at the point x_0 smaller than $C(\Sigma)$ the following property holds:*

$$\delta^2 \Phi_{00}(\xi) > 0 \text{ for all } \xi \perp v_0$$

and

$$\delta^2 \Phi_{00}(\xi) < 0 \text{ for all } \xi \parallel v_0$$

Proof. Follows immediately from the explicit formulas above. Indeed for φ small enough in (4) $(\frac{1}{L(x_{-1}, x_0)} + \frac{1}{L(x_0, x_1)})$ becomes large while $B(\xi, \xi) \hat{v}_0$ tends to zero. This gives the first inequality of the theorem. In order to prove the second, one needs to represent the surface near the point x_0 by a graph of a convex function and uses Taylor expansions near x_0 in (5). We omit the details, since they are the same as in a planar case. \square

The different behavior of the second variation for transversal and longitudinal perturbations in the last theorem was discussed on a different language of *fronts* by L.Bunimovich ([6]).

4. PROOF OF THE THEOREM 1.3, EXAMPLES

Theorem 1.3 is an immediate consequence of Theorem 3.1, because any segment of a locally maximizing orbit has to have second variation negative semi-definite.

Corollary 4.1. *Let $x \in \Sigma$ be a point with a "small" second fundamental form:*

$$k_\xi = B(\xi, \xi) < \frac{1}{D}, \text{ for all } \xi \in T_x \Sigma \text{ with } |\xi| = 1,$$

then no maximizing segment passes through x .

Proof. It follows from the identity (4) for $|\xi| = 1$

$$\delta^2 \Phi_{00}(\xi) = \left(\frac{1}{L(x_{-1}, x_0)} + \frac{1}{L(x_0, x_1)} \right) - 2B(\xi, \xi) \hat{v}_0 > \frac{2}{D} - \frac{2}{D} \sin \varphi_0 > 0.$$

\square

Remark 2. This corollary can be regarded as analogous to the construction of Riemannian metrics with "big bumps", where no minimal geodesics pass through the top of the bump, see [5],[1] for details. Let me mention also that no such construction can be invented for planar Birkhoff billiards where through any point on the boundary pass infinitely many maximizing orbits due to existence of caustics near the boundary.

Example 1. Consider for example Ellipsoid in \mathbb{R}^3

$$E = \left\{ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \right\}, \text{ for } a_1 \leq a_2 \leq a_3.$$

Consider the point $A = (a_1, 0, 0)$ on the shortest axes. Then the principle curvatures of A are $\frac{a_1}{a_2^2}, \frac{a_1}{a_3^2}$ and the diameter is $D = 2a_3$. Therefore, if $2a_1a_3 < a_2^2$, then there are no maximizing orbits passing through A .

Our next example is somewhat opposite to the previous one. It shows that for certain shapes there are in fact many infinite maximizing orbits.

Example 2. Let γ be a planar smooth strictly (of positive curvature) convex closed curve. Let Σ be smooth hypersurface in \mathbb{R}^3 containing γ and symmetric with respect to the plane containing γ . Then obviously any orbit of the planar billiard inside γ remains an orbit of the billiard inside Σ . Let me denote the principle curvatures of Σ at the points of γ by k_1 -in the direction of γ , and k_2 -in the orthogonal direction. Let $\{x_n\}_{n \in \mathbb{Z}}$ be an infinite Aubry-Mather orbit in the plane of γ . Then all the angles of reflections φ_n are bounded away from zero, i.e. there is $C_1 > 0$, so that $\sin \varphi_n > C_1$. We claim, that if k_2 is sufficiently large then the orbit $\{x_n\}_{n \in \mathbb{Z}}$ remains maximizing inside the billiard in Σ .

Proof. In order to prove the claim we shall examine the second variation. Any field ξ_n along the orbit $\{x_n\}$ can be spited into a sum of two, the first is in the direction of γ and the second is in the orthogonal direction, $\xi_n = \xi'_n + \xi''_n$. Then we have for the second variations

$$\delta^2\Phi(\xi) = \delta^2\Phi(\xi') + \delta^2\Phi(\xi'') + 2\delta^2\Phi(\xi', \xi'')$$

where the mixed term $2\delta^2\Phi(\xi', \xi'')$ vanishes, due to the symmetry assumptions. Since the orbit $\{x_n\}_{n \in \mathbb{Z}}$ is the Aubry-Mather orbit inside γ then it is maximizing with respect to planar perturbations and therefore $\delta^2\Phi(\xi') < 0$. Let us estimate the $\delta^2\Phi(\xi'')$. Denote by

$$K_1 = \max_{\gamma} k_1, \quad K_2 = \min_{\gamma} k_2.$$

For the planar billiard one has,

$$L(x_n, x_{n+1}) > 2 \sin \varphi_n / K_1 > 2C_1 / K_1, \text{ for all } n \in \mathbb{Z}.$$

Let a_n, b_n be the diagonal and the off-diagonal elements of the quadratic form $\delta^2\Phi(\xi'')$ respectively. Then

$$a_n = \left(\frac{1}{L(x_{n-1}, x_n)} + \frac{1}{L(x_n, x_{n+1})} \right) - 2k_2(x_n) \sin \varphi_n < K_1/C_1 - 2K_2 := a$$

Also

$$|b_n| = 1/L(x_n, x_{n+1}) < K_1/2C_1 := b.$$

Let the number K_2 be big enough so that $a < 0$ and $-a > 2b$. Taking into account all the estimations we obtain:

$$\delta^2\Phi_{MN}(\xi^{(2)}) = \sum_{n=M}^N a_n \xi_n''^2 + 2 \sum_{n=M}^{N-1} b_n \xi_n'' \xi_{n+1}'' < a \sum_{n=M}^N \xi_n''^2 + 2b \sum_{n=M}^{N-1} \xi_n'' \xi_{n+1}''$$

It is an easy exercise to see that the last expression is negative. This completes the proof. \square

Example 3. This is example of the standard sphere. In this example non of the orbits except diameters are maximizing. For instance all periodic orbits are not maximizing. Let me remark that they are maximizing for the variational principle on the set of closed n-gons. This shows the difference between the two variational principle. We refer the reader to recent paper ([7]) for the results on periodic trajectories in general and estimation of there number in higher dimension. Any orbit of the billiard inside the standard sphere is completely determined by the angle of reflection α (for sphere they are all the same). Computing the quadratic form of the second variation with respect to transversal perturbations gives the following matrix

$$\begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots b & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{pmatrix}, \text{ where } a = \frac{\cos 2\alpha}{\sin \alpha}, \quad b = -\frac{1}{2 \sin \alpha}$$

It is an easy exercise to show that for any angle α choosing the size of this matrix is big enough one gets the matrix which is not negative definite. Let me remark here that the quadratic form with respect to the longitudinal perturbations gives the same matrix with the parameters $a = -\sin \alpha, b = \sin \alpha/2$, which is negative definite.

5. CONJUGATE POINTS, PROOF OF THEOREM 1.2

Let me introduce some notations. For a fixed point $x \in \Sigma$ and $n \geq 1$ introduce the subset $M_{x,n}$ of $B_x^*\Sigma$ consisting of those (co)vectors v such that the corresponding billiard trajectory $\{x_0 = x, x_1, \dots, x_{n+1}\}$ is maximizing, that is $\delta^2\Phi_{1,n}$ is negative semi-definite. Obviously $M_{x,n}$ is not empty since given $x, y \in \Sigma$ one can set $x_0 = x, x_{n+1} = y$ and find $\{x_1, \dots, x_n\}$ giving the global maximum to the length functional $\Phi_{1,n}$.

Moreover $M_{x,n}$ is closed by the continuity of the quadratic form $\delta^2\Phi_{1,n}$. It follows from Theorem 1.3 that this set is compact in the open ball $B_x^*\Sigma$, because no maximizing orbits near the boundary of the ball are allowed. Now it is easy to prove Theorem 1.2

Proof. Let v be any vector lying in the boundary $\partial M_{x,n}$ and consider the orbit corresponding to (x, v) : $\{x_0 = x, x_1, \dots, x_{n+1}\}$. Then it follows that the quadratic form of this orbit $\delta^2\Phi_{1,n}$ is negative semi-definite and also has a nontrivial Kernel. Then there exists a field $\{\xi_1, \dots, \xi_n\}$ lying in the Kernel. Then it must satisfy the following equation:

$$(L_{22}(x_{k-1}, x_k) + L_{11}(x_k, x_{k+1}))\xi_k + L_{21}(x_{k-1}, x_k)\xi_{k-1} + L_{12}(x_k, x_{k+1})\xi_{k+1} = 0, \text{ for all } k = 1, \dots, n$$

This is precisely the equation of the Jacobi fields.

Thus $\{\xi_0 = 0, \xi_1, \dots, \xi_n, \xi_{n+1} = 0\}$ is a Jacobi field vanishing at the ends. Therefore, x_0, x_{n+1} are conjugate. This yields the proof. \square

Notice that $M_{x,n+1} \subseteq M_{x,n}$ and by the compactness it follows that their intersection is not empty $\bigcap M_{x,n} \neq \emptyset$.

Corollary 5.1. *For any point $x \in \Sigma$ there exists infinite semi-orbit $\{x_n\}_{n \geq 0}$, $x_0 = x$ starting at x which is a maximizing.*

Let me remark that the semi-orbit in the last Corollary is not claimed to be an infinite orbit, see the Example 1 where no infinite orbit passes through a certain point. It would be interesting to know any example different from bodies of constant width, where there are maximizing orbits passing through every point of Σ .

Remark 3. Let me explain, that the compacts $M_{x,n}$ are in fact rather "fat". Denote by $M'_{x,n}$ the set of all those $v \in B_x^*\Sigma$ such that the corresponding orbit $\{x_0 = x, x_1, \dots, x_{n+1}\}$ has negative definite quadratic form $\delta^2\Phi_{1,n}$. Obviously $M'_{x,n} \subseteq M_{x,n}$. Moreover the following important inclusion holds:

$$M'_{x,n+1} \subseteq M_{x,n},$$

which means that any proper subsegment of a maximizing segment has a non-degenerate second variation form. This is of course a well known fact in Riemannian case. It was proved for twist maps in ([9]) (in the case of higher dimensional billiards the twist condition is that the operators L_{12} and L_{21} are isomorphisms and the proof of ([9]) goes through with no change). We refer also to ([4]) for more discussions on the twist maps.

REFERENCES

- [1] Bangert, V. Mather sets for twist maps and geodesics on tori. Dynamics reported 1 (1987).

- [2] Berger, M. Sur les caustiques de surfaces en dimension 3. C. R. Acad. Sci. Paris Ser.I Math. 311 (1990), no. 6, 333-336.
- [3] Bialy, M. Convex billiards and a theorem by E. Hopf. Math. Z. 214 (1993), no. 1, 147-154.
- [4] Bialy, M., MacKay, R. Symplectic twist maps without conjugate points. Israel J. Math. 141 (2004), 235-247.
- [5] Bialy, M., Polterovich, L. Geodesic flows on the two-dimensional torus and phase transitions "commensurability-noncommensurability". Funct. Anal. Appl. 20, 260-266 (1986).
- [6] Bunimovich, L. Dispersing, Defocusing and Astigmatism. Math. Education 5 (2001) 106-124 (in Russian).
- [7] Farber, M., Tabachnikov, S. Topology of cyclic configuration spaces and periodic orbits of multi-dimensional billiards. Topology, 41 (2002), 553-589.
- [8] Lazutkin, V. Existence of caustics for the billiard problem in a convex domain. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 186-216.
- [9] MacKay, R., Meiss, J., Strark, J. Converse KAM theory for symplectic twist maps, Nonlinearity 2 (1989) 555-570.
- [10] Tabachnikov, S. Billiards, Panor. Synth. 1 (1995).

RAYMOND AND BEVERLY SACKLER SCHOOL OF MATHEMATICAL SCIENCES,
TEL AVIV UNIVERSITY, ISRAEL
E-mail address: bialy@post.tau.ac.il